

Geometric Phases for Coherent States

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We explore geometric phases of coherent states and some of their properties. A better and elegant expression of geometric phase for coherent state is derived. It is used to obtain the explicit form of the geometric phase for entangled coherent states, several interesting results followed by considering different cases for the parameters. The effects of entanglement and harmonic potential on the geometric phase are discussed.

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I. INTRODUCTION

The phase factor of a wave function is one of the most fundamental concepts in quantum physics. It is Berry who first discovered that a geometric phase can be accrued in the wave function of a quantum system in adiabatic, unitary, and cyclic evolution of time-dependent quantum system [1]. Three years later, by abandoning the context of adiabaticity, Berry's result was extended by Aharonov and Anandan [2]. Soon after Samuel and Bhandari [3] further generalized the Aharonov and Anandan phase to non-cyclic and non-unitary cases by resorting to Pancharatnam's pioneer work [4]. Subsequently, Ref. [5] established the quantum kinematic approach to geometric phases, which is the most general theory on geometric phases of pure quantum states. However the above definitions of geometric phases are not applicable, if the initial and final states are orthogonal. Manini and Pistolesi [6] first proposed a method to solve the problem by introducing the Abelian off-diagonal geometric phases during adiabatic evolution. One year later the definition was generalized to nonadiabatic cases with the use of Bargmann invariants [7]. Recently, Kult *et al.* [8] made a step forward in this direction by extending the concepts to non-Abelian systems. Despite all the results achieved for pure states in the literatures, the definitions of geometric phases have also been generalized to the case of mixed states. Uhlmann [9] presented a definition in mathematical context of purification. Sjöqvist *et al.* [10] developed the non-degenerate geometric phase in non-cyclic and unitary evolution under the background of quantum interference, which is independent of surroundings. Extensions of mixed-state geometric phases to the degenerate case [11] and the kinematic approach [12] have also been achieved.

The Berry phase and its generalizations have given

rise to many applications ranging from condensed matter physics [13] to quantum information science [14–17]. For example, geometric phase plays an important role in quantum information and computation protocols which employ dynamical or geometric phases to achieve quantum gates. Dynamical phase gates require a precise control of the pulse area. Geometric phases depend only on the solid angle enclosed by the parameter path and generally not on the details of the path. Thus geometric phases can render robust protocols for quantum computation [15–17]. The promising applications spawn various theoretical investigations on geometric phases of different physical systems [18–23]. Ref. [18] reported entanglement dependence of the non-cyclic and non-adiabatic geometric phase for entangled spin pairs in a static magnetic field, and the result was promoted to entangled spin particles in a rotating magnetic field later [19]. Literatures [21–23] explored the geometric phases of coherent states of a one-dimensional harmonic oscillator. Chaturvedi *et al.* found the Berry phase for coherent states [21]; the geometric phase for the noncyclic evolution of coherent states was studied in Ref. [22]; and the authors in [23] formulated the non-unitary and non-cyclic geometric phases for nonlinear coherent states.

In this work we study the geometric phases of coherent states, especially entangled coherent states. In the next section, we briefly review some concepts of kinematic approach to geometric phases. These ideas are illustrated by considering the geometric phase for coherent states of a one-dimensional harmonic oscillator. We find an equivalent but more elegant form of geometric phase than that given in Ref. [22]. In Sec. III, the geometric phases of entangled coherent states are calculated and some of their properties are discussed. We study the influences of entanglement and harmonic potential on the geometric phases. We end with conclusion in the last section.

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II. BRIEF REVIEW OF QUANTUM KINEMATIC APPROACH TO GEOMETRIC PHASE

In this section, the rudiment about the kinematic approach to the non-cyclic geometric phase [5] is reviewed. When a quantum system undergoes a unitary evolution, its state vectors trace a smooth trajectory in Hilbert space, which is $\mathcal{C} = \{|\psi(t)\rangle \in \mathcal{H} | t \in [0, \tau] \subset \mathcal{R}\}$. Accompanying the quantum evolution, there exists a geometric quantity that is both gauge invariant and reparametrization invariant, under the condition that $\langle\psi(0)|\psi(\tau)\rangle \neq 0$. Such a geometric quantity is called geometric phase, which is expressed as follows

$$\gamma = \chi - \delta, \quad (1)$$

where χ is total phase

$$\chi = \arg(\langle\psi(0)|\psi(\tau)\rangle), \quad (2)$$

and δ is dynamical phase

$$\delta = -i \int_{t=0}^{t=\tau} \langle\psi(t)| \frac{d}{dt} |\psi(t)\rangle. \quad (3)$$

In order to illustrate the above definition clearly, we consider a one-dimensional harmonic oscillator whose Hamiltonian reads

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

and the state vector at any later time τ is given by

$$|\alpha, \tau\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega\tau(n+1/2)} |n\rangle. \quad (4)$$

where α is a complex number representing the coherent state and n labels number state. According to Eq. (2), the total phase is

$$\chi = \arg(\langle\alpha, 0|\alpha, \tau\rangle) = -(|\alpha|^2 \sin \omega\tau + \frac{1}{2}\omega\tau), \quad (5)$$

where we have used the inner product $\langle\alpha, 0|\alpha, \tau\rangle = e^{-|\alpha|^2(1-\cos \omega\tau)} e^{-i(|\alpha|^2 \sin \omega\tau + \frac{1}{2}\omega\tau)}$. By using Eq. (3), the dynamical phase is equal to

$$\delta(\alpha) = -\omega\tau(\frac{1}{2} + |\alpha|^2). \quad (6)$$

Substituting Eq. (5) and (6) into Eq. (1), one obtains the corresponding geometric phase that takes the form

$$\gamma = |\alpha|^2(\omega\tau - \sin \omega\tau). \quad (7)$$

Let us point out that the geometric phase (7) is an elegant result than that of Ref. [22]. When $\omega\tau = 2\pi$, the known cyclic geometric phase

$$\gamma = |\alpha|^2\omega\tau \quad (8)$$

is recovered [22].

III. GEOMETRIC PHASES FOR ENTANGLED COHERENT STATES

A. Entangled coherent states of harmonic oscillators

Pertaining to harmonic oscillators, a general unnormalized two-particle entangled coherent state [24] is of the form

$$|\Psi(t)\rangle = e^{-i\varphi/2} \cos \frac{\theta}{2} |\alpha, t\rangle_1 |\mu, t\rangle_2 + e^{i\varphi/2} \sin \frac{\theta}{2} |\beta, t\rangle_1 |\nu, t\rangle_2, \quad (9)$$

where α, β, μ and ν are complex parameters of the corresponding coherent states (4), the subscripts denote particle 1 and 2 respectively, and θ as well as φ are real numbers. θ is a quantity which determines the degree of entanglement of the two-particle entangled coherent state. When $\theta = 0$ or $\theta = \pi$, the entangled coherent state (9) reduces to product state; when $\theta = \pi/2$, the degree of entangled coherent state becomes maximal. Note that the coherent state vectors of subsystems are generally not orthogonal to each other:

$$\langle\alpha, t|\beta, t\rangle = \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta], \quad (10)$$

which is quite different from the case of entangled spin-1/2 pairs [18]. The evolutionary state studied in [18] is given by

$$|\Psi(t)\rangle = e^{-i\varphi/2} \cos \frac{\theta}{2} |+\mathbf{n}, t\rangle_1 |+\mathbf{m}, t\rangle_2 + e^{i\varphi/2} \sin \frac{\theta}{2} |-\mathbf{n}, t\rangle_1 |-\mathbf{m}, t\rangle_2, \quad (11)$$

where $|+\mathbf{n}, t\rangle_1$ (or $|+\mathbf{m}, t\rangle_2$) and $|-\mathbf{n}, t\rangle_1$ (or $|-\mathbf{m}, t\rangle_2$) are mutually orthogonal. Hence it is reasonable to expect that the total phase, dynamical phase and geometric phase of entangled coherent states will exhibit some different features from that of entangled spin pairs as given in Ref. [18]. For the sake of completeness, we write down the normalized entangled coherent state in the following

$$|\psi(t)\rangle = \frac{1}{\mathcal{N}} (e^{-i\varphi/2} \cos \frac{\theta}{2} |\alpha, t\rangle_1 |\mu, t\rangle_2 + e^{i\varphi/2} \sin \frac{\theta}{2} |\beta, t\rangle_1 |\nu, t\rangle_2), \quad (12)$$

where the normalization factor satisfies

$$\mathcal{N}^2 = 1 + \sin \theta \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) - \frac{1}{2}(|\mu|^2 + |\nu|^2)] \times \text{Re}[\exp(i\varphi + \alpha^*\beta + \mu^*\nu)], \quad (13)$$

which is independent of time and for simplicity we choose \mathcal{N} as a positive real number.

B. Geometric phases for entangled coherent states

Consider a system that consists of two non-interacting entangled coherent particles which are in harmonic po-

tentials. The corresponding Hamiltonian reads

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}k_1x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}k_2x_2^2. \quad (14)$$

Because there is not interaction between the two particles, its time evolution operator takes a product form. If we choose initial state as

$$|\psi(0)\rangle = \frac{1}{\mathcal{N}}(e^{-i\varphi/2} \cos \frac{\theta}{2} |\alpha, 0\rangle_1 |\mu, 0\rangle_2 + e^{i\varphi/2} \sin \frac{\theta}{2} |\beta, 0\rangle_1 |\nu, 0\rangle_2), \quad (15)$$

then at any later time $t > 0$, the state vector can be written in the form as shown in Eq. (12).

In order to calculate the geometric phases of this system, we need to determine its total phase and dynamical phase respectively. By the use of the formula below

$$\langle \alpha, 0 | \beta, \tau \rangle = \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^* \beta e^{-i\omega\tau} - i\frac{1}{2}\omega\tau] \quad (16)$$

we get

$$\mathcal{N}^2 \langle \psi(0) | \psi(\tau) \rangle = \frac{1+\cos\theta}{2} f(\alpha; \mu) \exp[i\chi(\alpha; \mu)] + \frac{1-\cos\theta}{2} f(\beta; \nu) \exp[i\chi(\beta; \nu)] + \frac{1}{2} \sin\theta g(\alpha, \beta; \mu, \nu) \exp\{i[h(\alpha, \beta; \mu, \nu) + \varphi]\} + \frac{1}{2} \sin\theta g(\beta, \alpha; \mu, \nu) \exp\{i[h(\beta, \alpha; \mu, \nu) - \varphi]\}, \quad (17)$$

where

$$\begin{aligned} f(\alpha; \mu) &= \exp[-\rho_\alpha^2(1 - \cos\omega_1\tau) - \rho_\mu^2(1 - \cos\omega_2\tau)], \\ g(\alpha, \beta; \mu, \nu) &= \exp[-\frac{1}{2}(\rho_\alpha^2 + \rho_\beta^2) - \frac{1}{2}(\rho_\mu^2 + \rho_\nu^2) + \rho_\alpha\rho_\beta \cos(\phi_\alpha - \phi_\beta + \omega_1\tau) + \rho_\mu\rho_\nu \cos(\phi_\mu - \phi_\nu + \omega_2\tau)], \\ \chi(\alpha; \mu) &= -(\rho_\alpha^2 \sin\omega_1\tau + \frac{1}{2}\omega_1\tau) - (\rho_\mu^2 \sin\omega_2\tau + \frac{1}{2}\omega_2\tau), \\ h(\alpha, \beta; \mu, \nu) &= -[\rho_\alpha\rho_\beta \sin(\phi_\alpha - \phi_\beta + \omega_1\tau) + \frac{1}{2}\omega_1\tau] - [\rho_\mu\rho_\nu \sin(\phi_\mu - \phi_\nu + \omega_2\tau) + \frac{1}{2}\omega_2\tau], \end{aligned} \quad (18)$$

with ρ_λ and ϕ_λ being the amplitude and phase of the state $|\lambda\rangle$ and $\lambda = \rho_\lambda e^{i\phi_\lambda}$ ($\lambda = \alpha, \beta, \mu, \nu$). Hence, the total phase is found to be

$$\chi(\alpha, \beta; \mu, \nu) = \arg(\langle \psi(0) | \psi(\tau) \rangle) = \arctan \frac{A}{B}, \quad (19)$$

where

$$\begin{aligned} A &= \text{Im}(\langle \psi(0) | \psi(\tau) \rangle) \\ &= (1 + \cos\theta) f(\alpha; \mu) \sin[\chi(\alpha; \mu)] + (1 - \cos\theta) f(\beta; \nu) \sin[\chi(\beta; \nu)] \\ &\quad + \sin\theta g(\alpha, \beta; \mu, \nu) \sin[h(\alpha, \beta; \mu, \nu) + \varphi] + \sin\theta g(\beta, \alpha; \mu, \nu) \sin[h(\beta, \alpha; \mu, \nu) - \varphi], \end{aligned} \quad (20)$$

and

$$\begin{aligned} B &= \text{Re}(\langle \psi(0) | \psi(\tau) \rangle) \\ &= (1 + \cos\theta) f(\alpha; \mu) \cos[\chi(\alpha; \mu)] + (1 - \cos\theta) f(\beta; \nu) \cos[\chi(\beta; \nu)] \\ &\quad + \sin\theta g(\alpha, \beta; \mu, \nu) \cos[h(\alpha, \beta; \mu, \nu) + \varphi] + \sin\theta g(\beta, \alpha; \mu, \nu) \cos[h(\beta, \alpha; \mu, \nu) - \varphi]. \end{aligned} \quad (21)$$

In the following, the dynamical phase is determined by using the formula

$$\langle \alpha, t | \frac{d}{dt} | \beta, t \rangle = -i\omega \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) - \alpha^* \beta] (\frac{1}{2} + \alpha^* \beta). \quad (22)$$

The dynamical phase (3) is then found to be

$$\begin{aligned} \delta(\alpha, \beta; \mu, \nu) &= -\frac{1}{\mathcal{N}^2} \frac{1+\cos\theta}{2} [\omega_1\tau(\frac{1}{2} + |\alpha|^2) + \omega_2\tau(\frac{1}{2} + |\mu|^2)] - \frac{1}{\mathcal{N}^2} \frac{1-\cos\theta}{2} [\omega_1\tau(\frac{1}{2} + |\beta|^2) \\ &\quad + \omega_2\tau(\frac{1}{2} + |\nu|^2)] - \frac{1}{\mathcal{N}^2} \text{Re}\{\sin\theta e^{i\varphi} \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^* \beta] \\ &\quad - \frac{1}{2}(|\mu|^2 + |\nu|^2) + \mu^* \nu][\omega_1\tau(\frac{1}{2} + \alpha^* \beta) + \omega_2\tau(\frac{1}{2} + \mu^* \nu)]\}. \end{aligned} \quad (23)$$

According to Eq. (1), we can obtain the geometric phase as

$$\gamma(\alpha, \beta; \mu, \nu) = \chi(\alpha, \beta; \mu, \nu) - \delta(\alpha, \beta; \mu, \nu). \quad (24)$$

However, the above expression is too tedious to use. So we concentrate on a particular type of entangled coherent states by setting $\beta = -\alpha$ and $\nu = -\mu$, and thus the

geometric phase reads

$$\gamma(\alpha, -\alpha; \mu, -\mu) = \arctan \left\{ \frac{f(\alpha; \mu) \sin[\chi(\alpha; \mu)] + \sin \theta \cos \varphi g(\alpha, -\alpha; \mu, -\mu) \sin[h(\alpha, -\alpha; \mu, -\mu)]}{f(\alpha; \mu) \cos[\chi(\alpha; \mu)] + \sin \theta \cos \varphi g(\alpha, -\alpha; \mu, -\mu) \cos[h(\alpha, -\alpha; \mu, -\mu)]} \right\} \\ + \frac{1}{1 + \sin \theta \cos \varphi} \left\{ \omega_1 \tau \left(\frac{1}{2} + \rho_\alpha^2 \right) + \omega_2 \tau \left(\frac{1}{2} + \rho_\mu^2 \right) + \sin \theta \cos \varphi \exp[-2(\rho_\alpha^2 + \rho_\mu^2)] [\omega_1 \tau \left(\frac{1}{2} - \rho_\alpha^2 \right) + \omega_2 \tau \left(\frac{1}{2} - \rho_\mu^2 \right)] \right\}. \quad (25)$$

C. Discussions

The following discussions are confined to Eq. (25). Namely we explore the properties of the geometric phase for the specific type of entangled coherent state with $\beta = -\alpha$ and $\nu = -\mu$.

Case 1. First we examine the effect of entanglement on the geometric phase. Consider $\theta = 0$ or $\theta = \pi$, and this leads to vanishing entanglement, or product coherent states. Therefore, the geometric phase is

$$\begin{aligned} \gamma(\alpha; \mu) &= \gamma(-\alpha; -\mu) \\ &= |\alpha|^2 (\omega_1 \tau - \sin \omega_1 \tau) + |\mu|^2 (\omega_2 \tau - \sin \omega_2 \tau), \end{aligned} \quad (26)$$

It is clear that the geometric phase is the exact addition of geometric phases (7) acquired by either subsystem. Our result shows that the geometric phase of any product coherent state is equal to the sum of geometric phases acquired by either subsystem. It is not difficult to find that only when $|\alpha|^2 \rightarrow \infty$ and $|\mu|^2 \rightarrow \infty$, the total phase

vanishes. So under the case of finite valued parameters, it is not necessary to study the corresponding off-diagonal phase [6, 7]. Under the condition that $\theta = \pi/2$, the entanglement of this state becomes maximal. Unlike the case of entangled spin pairs [18], the dynamical phase still exists and $\langle \psi(0) | \psi(\tau) \rangle$ is not a real number, and hence the geometric phase (25) also emerges.

Case 2. Let us look at cyclic two-particle geometric phase from Eq. (25). When τ satisfies the following conditions

$$\begin{cases} \omega_1 \tau = 2\pi l_1 \\ \omega_2 \tau = 2\pi l_2 \end{cases},$$

where l_1 and l_2 are integers, the corresponding wave function takes the form

$$|\psi(\tau)\rangle = e^{-i(\pi l_1 + \pi l_2)} |\psi(0)\rangle, \quad (27)$$

which is a cyclic evolution [2]. Then we obtain cyclic geometric phase for entangled coherent states,

$$\begin{aligned} \gamma^c &= -(\pi l_1 + \pi l_2) + 2\pi \frac{l_1(\frac{1}{2} + \rho_\alpha^2) + l_2(\frac{1}{2} + \rho_\mu^2) + \sin \theta \cos \varphi \exp[-2(\rho_\alpha^2 + \rho_\mu^2)] [l_1(\frac{1}{2} - \rho_\alpha^2) + l_2(\frac{1}{2} - \rho_\mu^2)]}{1 + \sin \theta \cos \varphi}, \\ &= \gamma_1^c + \gamma_2^c \end{aligned} \quad (28)$$

where γ_1^c is given in Eq. (30) and γ_2^c takes a similar form. Similar to the result of entangled spin pairs in Ref. [19], when the system of entangled coherent states undergoes cyclic evolutions (27), the total geometric phase is equal to the sum of all subsystem, no matter whether entan-

glement exists or not.

Case 3. We then consider how harmonic potential affects the geometric phases. Assume that only particle 1 is in the harmonic potential, and the geometric phase of particle 1 is found to be

$$\arctan \left\{ \frac{\exp[-\rho_\alpha^2(1 - \cos \omega_1 \tau)] \sin[-(\rho_\alpha^2 \sin \omega_1 \tau + \frac{1}{2} \omega_1 \tau)] + \sin \theta \cos \varphi \exp[-\rho_\alpha^2(1 + \cos \omega_1 \tau) - 2\rho_\mu^2] \sin(\rho_\alpha^2 \sin \omega_1 \tau - \frac{1}{2} \omega_1 \tau)}{\exp[-\rho_\alpha^2(1 - \cos \omega_1 \tau)] \cos[-(\rho_\alpha^2 \sin \omega_1 \tau + \frac{1}{2} \omega_1 \tau)] + \sin \theta \cos \varphi \exp[-\rho_\alpha^2(1 + \cos \omega_1 \tau) - 2\rho_\mu^2] \cos(\rho_\alpha^2 \sin \omega_1 \tau - \frac{1}{2} \omega_1 \tau)} \right\} - \delta_1 \quad (29)$$

via setting $\omega_2 = 0$, where the one-particle dynamical phase δ_1 is

$$\delta_1 = - \frac{\omega_1 \tau (\frac{1}{2} + \rho_\alpha^2) + \sin \theta \cos \varphi \exp[-2(\rho_\alpha^2 + \rho_\mu^2)] [\omega_1 \tau (\frac{1}{2} - \rho_\alpha^2)]}{1 + \sin \theta \cos \varphi}.$$

Though particle 2 does not experience harmonic potential, it also has an influence on the geometric phase of particle 1 through the entanglement between the two particles. By setting $\theta = 0$ or π , the geometric phase (29)

reduces to Eq. (7). This is due to the fact that the two particles are not entangled in this case. When the system undertakes a cyclic evolution, i.e., $\omega_1\tau = 2\pi l_1$, we obtain the cyclic one-particle geometric phase which is

$$\gamma_1^c = -\pi l_1 + \frac{2\pi}{1 + \sin\theta \cos\varphi} \{l_1(\frac{1}{2} + \rho_\alpha^2) + l_1(\frac{1}{2} - \rho_\alpha^2) \sin\theta \cos\varphi \exp[-2(\rho_\alpha^2 + \rho_\mu^2)]\}. \quad (30)$$

We also find two-particle dynamical phase as

$$\delta = -\frac{\omega_1\tau(\frac{1}{2} + \rho_\alpha^2) + \omega_2\tau(\frac{1}{2} + \rho_\mu^2) + \sin\theta \cos\varphi \exp[-2(\rho_\alpha^2 + \rho_\mu^2)][\omega_1\tau(\frac{1}{2} - \rho_\alpha^2) + \omega_2\tau(\frac{1}{2} - \rho_\mu^2)]}{1 + \sin\theta \cos\varphi}. \quad (31)$$

It is reasonable to draw a conclusion that the total dynamical phase is identical with sum of counterparts of subsystems which is partially affected by the potential. It is worth to point out that the result is in accordance with that for spin particles as given in Ref. [19].

IV. CONCLUSION

We have derived an elegant expression of geometric phase for coherent states. The non-cyclic geometric phases for entangled coherent states and some of their properties are investigated. We explore the influences of entanglement and potential on the geometric phases. Our results show that when entanglement vanishes, the geometric phase of product state is equal to the sum of counterparts of individual particles. In the case of maximally entangled coherent states, the dynamical phase and geometric phase do not disappear, and the result is different from that for entangled spin pairs. We also explore the property of cyclic two-particle geometric phases. It is found that the cyclic geometric phase is identical with addition of counterparts of subsystems no matter whether

entanglement vanishes or not. Our findings are consistent with the results for entangled spin pairs. If only one particle is affected by harmonic potential, both un-cyclic and cyclic one-particle geometric phases are affected by the other particle due to the existence of entanglement between the two particles. It is worth to mention that two-particle dynamical phase is not equal to the addition of counterparts of particles 1 and 2, while it can be expressed in terms of $\delta_1 + \delta_2$, where δ_1/δ_2 is cyclic dynamical phase of particle 1/2 experiencing harmonic potential while no potential acting on particle 2/1 respectively.

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- [1] M. V. Berry, *Proc. R. Soc. London, Ser. A* **392**, 45 (1984).
 - [2] A. Anandan and Y. Aharonov, *Phys. Rev. Lett.* **58**, 1593 (1987).
 - [3] J. Samuel and R. Bhandari, *Phys. Rev. Lett.* **60**, 2339 (1988).
 - [4] S. Pancharatnam, *Proc. Indian Acad. Sci., Sect. A* **43**, 247 (1956).
 - [5] N. Mukunda and R. Simon, *Ann. Phys.* **228**, 205 (1993).
 - [6] N. Manini and F. Pistolesi, *Phys. Rev. Lett.* **85**, 3067 (2000).
 - [7] N. Mukunda, Arvind, S. Chaturvedi, and R. Simon, *Phys. Rev. A* **65**, 012102 (2001).
 - [8] D. Kult, *Europhys. Lett.* **78**, 60004 (2007).
 - [9] A. Uhlmann, *Rep. Math. Phys.* **24**, 229 (1986).
 - [10] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, *Phys. Rev. Lett.* **85**, 2845 (2000).
 - [11] K. Singh, D. M. Tong, K. Basu, J. L. Chen, and J. F. Du, *Phys. Rev. A* **67**, 032106 (2003).
 - [12] D. M. Tong, E. Sjöqvist, L. C. Kwek, and C. H. Oh, *Phys. Rev. Lett.* **93**, 080405 (2004).
 - [13] D. J. Thouless, P. Ao, and Q. Niu, *Phys. Rev. Lett.* **76**, 3758 (1996); Q. H. Wang, *Phys. Rev. Lett.* **92**, 057003 (2004).
 - [14] L. M. Duan, J. I. Cirac, and P. Zoller, *Science* **292**, 1695 (2001).
 - [15] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, *Nature* **403**, 869 (2000).
 - [16] S. L. Zhu and Z. D. Wang, *Phys. Rev. Lett.* **91**, 187902 (2003).

- (2003).
- [17] X. L. Feng, Z. S. Wang, C. F. Wu, L. C. Kwek, C. H. Lai, and C. H. Oh, *Phys. Rev. A* **75**, 052312 (2007); C. F. Wu, Z. S. Wang, X. L. Feng, H. S. Goan, L. C. Kwek, and C. H. Oh, *Phys. Rev. A* **76**, 024302 (2007); X. L. Feng, C. F. Wu, H. Sun, and C. H. Oh, *Phys. Rev. Lett.* **103**, 200501 (2009).
 - [18] E. Sjöqvist, *Phys. Rev. A* **62**, 22109 (2000).
 - [19] D. M. Tong, L.C. Kwek, and C. H. Oh, *J. Phys. A: Math. and Gen.* **36**, 1149 (2003).
 - [20] D.B. Yang and J.L. Chen, *Commun. Theor. Phys.* **56**, 247 (2011).
 - [21] S. Chaturvedi, M. S. Sriram, and V. Srinivasan, *J. Phys. A: Math. and Gen.* **20**, L1071 (1987).
 - [22] A. K. Pati, *Phys. Rev. A* **52**, 2576 (1995).
 - [23] D. B. Yang, Y. Chen, F. L. Zhang, and J. L. Chen, *J. Phys. B: At. Mol. Opt. Phys* **44**, 075502 (2011).
 - [24] B. Wielinga and B. C. Sanders, *J. Mod. Opt.* **40**, 1923 (1993).